Enrollment No: ______________________ Exam Seat No: ______________________

C.U.SHAH UNIVERSITY

Winter Examination-2018

Subject Name: Number Theory

Subject Code: 5SC04NUT1 **Branch: M.Sc. (Mathematics)**

Semester:4 **Date: 20/10/2018** **Time:10:30 To 01:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the following questions** **(07)**

a. Define: Greatest integer function with example. **(01)**
b. Findgcd$(525, 231)$. **(01)**
c. Define: Prime number. **(01)**
d. Define: Multiplicative function. **(01)**
e. Congruence is an equivalent relation. (True/False) **(01)**
f. Number a is said to be a perfect number if $\sigma(a) = 2a$. (True/False) **(01)**
g. If k is any positive integer, then $k^2 + k + 1$ is a square number. (True/False) **(01)**

Q-2 **Attempt all questions** **(14)**

a. Prove that given any integers a and b, with $b > 0$, there exist unique integers q and r satisfying $a = qb + r, 0 \leq r < b$. **(05)**
b. In usual notations prove that, $[a, b](a, b) = ab$. **(05)**
c. Prove that the function τ and σ are both multiplicative functions. **(04)**

OR

Q-2 **Attempt all questions** **(14)**

a. Prove that every positive integer greater than one can be express uniquely as a product of prime, up to the order of the factor. **(05)**
b. If $2^k - 1$ is prime$(k > 1)$, then prove that $n = 2^{k-1}(2^k - 1)$ is perfect and every even perfect number is of the form. **(05)**
c. Let $N = a_0 + a_1 10 + a_2 10^2 + \cdots + a_m 10^m$ be the decimal expansion of the positive integer $N, 0 \leq a_k < 10$, and let $T = a_0 - a_1 + a_2 - \cdots + (-1)^m a_m$. Then prove that $11|N$ if and only if $11|T$. **(04)**

Q-3 **Attempt all questions** **(14)**

a. Prove that Mobious function is multiplicative function. **(05)**
b. State and prove Chinese remainder theorem. **(05)**
c. Prove that if p is prime and p/ab, then p/aor p/b. **(02)**
d. Prove that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then $a + c \equiv b + c \ (mod\ n)$ and $ac \equiv bc \ (mod\ n)$. **(02)**

OR

Q-3 **Attempt all questions** **(14)**

a. Solve: $6x \equiv 15\ (mod\ 21)$. **(05)**

b. Solve the system of congruences **(05)**

$$x \equiv 5(mod\ 6), x\ \equiv 4(mod\ 11), x \equiv 3(mod\ 17)$$

c. For integers a, b, c, if $c|a\ \&\ c|b$, then prove that $c|(ma + nb)$, $\forall\ m, n \in \mathbb{Z}$. **(02)**

d. Find highest power of 2 that divides$50!$. **(02)**

SECTION – II

Q-4 **Attempt the following questions** **(07)**

a. Define: Algebraic number. **(01)**

b. Write Pell's equation. **(01)**

c. Define: Primitive root. **(01)**

d. Find $\phi(9)$. **(01)**

e. State Fermat's Last Theorem. **(01)**

f. The product of two primitive polynomial is primitive. (True/False) **(01)**

g. Every Euclidean quadratic field has unique factorization property. (True/False) **(01)**

Q-5 **Attempt all questions** **(14)**

a. Prove that the integer p is prime if and only if $(p-1)! + 1 \equiv 0\ (mod\ p)$. **(05)**

b. Prove that the value of any infinite continued fraction is an irrational number. **(05)**

c. Express the rational number $\frac{19}{51}$ in finite simple continue fraction. **(02)**

d. Determine the infinite continued fraction representation of$\sqrt{7}$. **(02)**

OR

Q-5 **Attempt all questions** **(14)**

a. Let n be a positive rational integer and ξ a complex number. Suppose that the complex numbers $\theta_1, \theta_2, \theta_3, \dots, \theta_n$, not all zero, satisfy the equation **(05)**

$$\xi\theta_j\ = a_{j,1}\theta_1 + a_{j,2}\theta_2 + a_{j,n}\theta_n, \qquad j = 1,2,3, \dots$$

where the n^2 coefficients $a_{j,i}$are rational. Then prove that ξ is an algebraic number. Moreover, if the $a_{j,i}$ are rational integers, ξ ia an algebraic integer.

b. Solve the linear Diophantine equation $18\ x + 5y = 54$. **(05)**

c. If the integer a has order k modulo n, then prove that $a^i \equiv a^j\ (mod\ n)$ if and only if $i \equiv j(mod\ n)$. **(02)**

d. If p, q is a positive solution of $x^2 - dy^2 = 1$, then prove that $\frac{p}{q}$ is a convergent of the continued fraction expansion of$\sqrt{d}$. **(02)**

Q-6 **Attempt all questions** **(14)**

a. Prove that all the solutions of $x^2 + y^2 = z^2$with $x, y, z > 0$; satisfying the conditions $(x, y, z) = 1$, $2|x$are given by the formula **(05)**

$x = 2st, y = s^2 - t^2,\ z = s^2 + t^2$,where$s > t > 0, (s, t) = 1$ and one of s, t is even and the other is odd.

b. Determine the infinite continued fraction representation of irrational number$\sqrt{23}$. **(05)**

c. Compute the convergents of the simple continued fraction [1; 2,3,3,2,1]. **(04)**

OR

Q-6 **Attempt all questions** **(14)**

a. Define: Norm $N(\alpha)$in $Q(\sqrt{m})$. Prove that the norm of a product equals the product of the norms in $Q(\sqrt{m})$. **(05)**

b. If $C_k = \frac{p_k}{q_k}$ is the k^{th} convergent of finite simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$, then prove that $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, 1 \leq k \leq n$. **(05)**

c. Prove that if a monic polynomial $f(x)$ with integral coefficients factors in to two monic polynomails with rational coefficients, say $f(x) = g(x)h(x)$, then $g(x)$ and $h(x)$ have integral coefficients. **(04)**